UNIT - III

Sources, Sinks and Doublets (Three - dimensional Hydrodynamical Singularities)

Source: An outward Symmetrical radial flow in all directions is termed as a three dimensional Source or a Simple Source.

(or)

Suppose at a point O in a fluid the flow is Such that it is directed radially outwards from O in all directions and in a Symmetrical manner. Then fluid enters the Systems through O which is termed a Simple Source.

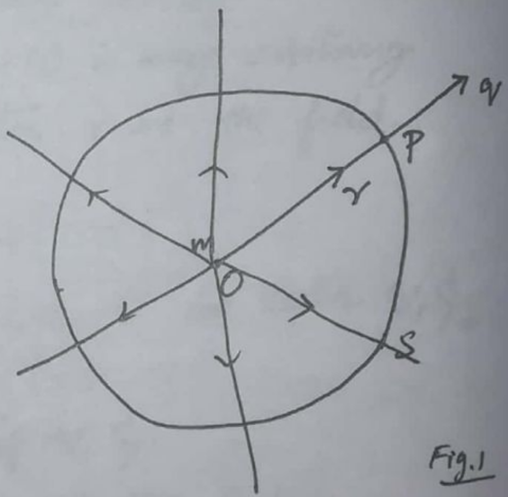


Fig.1

Sink: If at O the volume entering per unit time is $4\pi m$, where m is a constant, then the Strength of the Source is defined to be m.

If however, the flow is such that fluid is directed radially inwards to O from all directions in a Symmetric manner, then fluid leaves the System at O which is termed a Simple Sink.

A Sink of Strength m is a Source of Strength $-m$.

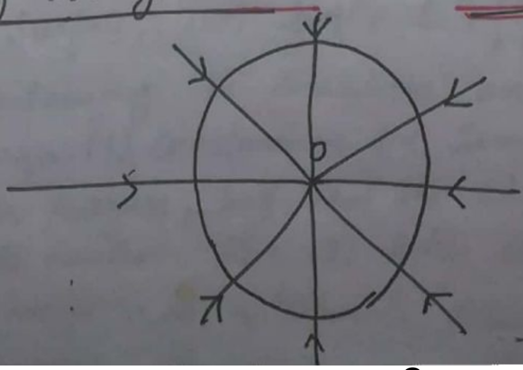


Fig.2

Fig. 1. Shows a
Simple source of strength m at O in a fluid which is presumed to contain no other sources or sinks and which would otherwise be at rest.

B.W.: Velocity of a fluid particle at due to source of strength m'.

S is the surface of the sphere centre, O and radius r and P is a field on S such that $\overline{OP} \equiv r$.

Then the fluid velocity at P is $\bar{q}$ along $\overline{OP}$ and magnitude q is everywhere constant over S.

The volume of fluid crossing S per unit time is $4\pi r^2 q$;

that emitted from O per unit time is $4\pi m$.

$\because$ the fluid is incompressible, these two are equal from considerations of continuity.

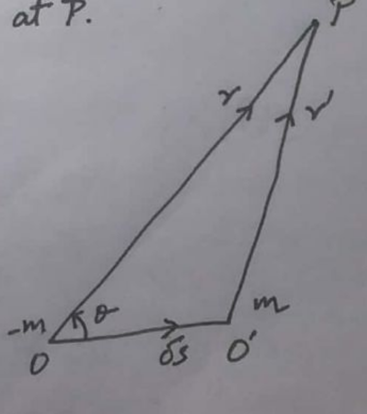
$$4\pi r^2 q = 4\pi m$$

$$\boxed{q = \frac{m}{r^2}} \quad \text{(Sink velocity)}$$

$$\bar{q} = \frac{m}{r^2}\hat{r} \quad \text{(in vector form)} \longrightarrow (1)$$

It is easily shown that $\nabla \times \bar{q} = 0$ (except at $r=0$) so that flow is of the potential kind.

Let ϕ be the velocity potential at P.



Let $\phi = \phi(r)$,

$$\nabla\phi = \phi'(r)\hat{r} \longrightarrow (2)$$

$$\text{Thus} \quad \bar{q} = -\nabla\phi \Rightarrow \nabla\phi = -\bar{q} \longrightarrow (3)$$

From (1) & (2), using 3,

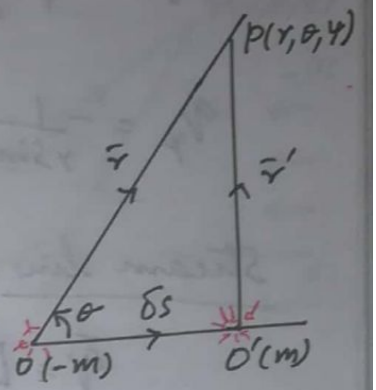
$$\therefore \phi'(r) = -\frac{m}{r^2}$$

Integrating,

$$\boxed{\phi(r) = -m\left(-\frac{1}{r}\right) = \frac{m}{r}}$$

Book work: Velocity potential due to doublet at $p(r,\theta,\psi)$

The velocity potential ϕ at $p(r,\theta,\psi)$ due to the doublet at O.



Let, the flow to be entirely due to $-m$ at O and m at O'.

The velocity Potential at P is $\phi = \frac{-m}{OP} + \frac{m}{O'P}$

$$\phi = \frac{m}{r'} - \frac{m}{r} = \frac{m(r-r')}{rr'}$$

$(\because \overline{OP} - \overline{O'P} = \overline{OO'}$, $r - r' = \delta s$

$$= \frac{m \cdot \delta s}{rr'} \cdot \frac{r+r'}{r+r'}$$

$$= m\,\delta s \left(\frac{r+r'}{rr'(r+r')} \right)$$

$$= \mu \left(\frac{r+r'}{rr'(r+r')} \right)$$

$\therefore m\delta s = \mu$

Let O – fixed

$O' \longrightarrow O$

$r' \longrightarrow r$

$\mu \longrightarrow$ constant. Then in the limit we have

$$\phi = \mu \left(\frac{\cancel{2r}}{r \cdot r(\cancel{2r})} \right) = \mu \cdot r^{-2} = (\mu \cdot r)\, r^{-3}$$

with $\angle POO' = \theta$,

$$\phi = (\mu \cdot \hat{r}) \cdot r^{-2}$$ (other equivalent forms for ϕ)

$$= \frac{|\mu|\,|\hat{r}|\,|\cos\theta|}{r^2}$$ $(\because |\hat{r}| = 1)$

$$= \boxed{\phi = \frac{\mu \cdot \cos\theta}{r^2}}$$

The velocity components at P, are

$$q_r = -\frac{\partial \phi}{\partial r} = -\mu\cos\theta\left(-\frac{2}{r^3}\right) = \frac{2\mu\cos\theta}{r^3}$$

$$q_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{-1}{r}\left(-\frac{M\sin\theta}{r^2}\right) = \frac{M\sin\theta}{r^3}$$

$$q_\psi = -\frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \psi} = \frac{-1}{r\sin\theta}(0) = 0$$

<u>Stream line:</u>

$$\bar{q} = [q_r, q_\theta, q_\psi] = \left[\frac{2\mu\cos\theta}{r^3}, \frac{M\sin\theta}{r^3}, 0\right]$$

The equation of stream lines are

$$\frac{dr}{q_r} = \frac{r\,d\theta}{q_\theta} = \frac{r\sin\theta\, dr}{q_\psi}$$

$$\frac{dr}{\frac{2\mu\cos\theta}{r^3}} = \frac{r\,d\theta}{\frac{M\sin\theta}{r^3}} = \frac{r\sin\theta\, d\psi}{0}$$

Comparing ratio I & II,

$$\frac{dr}{\frac{2\cancel{M}\cos\theta}{\cancel{r^3}}} = \frac{r\,d\theta}{\frac{\cancel{M}\sin\theta}{\cancel{r^3}}}$$

$$\frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}d\theta \Rightarrow \frac{dr}{r} = 2\cot\theta\, d\theta$$

Integrating,

$$\log r = 2\log\sin\theta + \log c$$

$$= \log\sin^2\theta + \log c$$

$$\log r = \log c\sin^2\theta$$

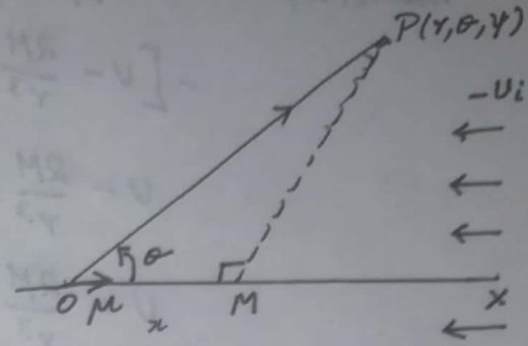
$$\boxed{r = c\sin^2\theta}$$

III ratio, $d\psi = 0$

Integrating $\boxed{\psi = \text{Constant}}$ $(\because \text{denominator} = 0)$

1) Doublet in an uniform stream

Figure shows a doublet of vector moment $\mu = \mu_i$ at O in a uniform stream.



Velocity (in the absence of the doublet) $= -Ui$

Find the velocity potential $\phi(r,\theta)$ at $P(r,\theta,\psi)$ in the fluid

Spherical Polar Coordinates

If PM $\perp^r$ from P on OX

if $OM = x$, then the velocity potential at P due to the streamline $Ux = Ur\cos\theta$

Due to the doublet at $O = \dfrac{\mu\cos\theta}{r^2}$

$\therefore$ The total velocity potential at P is

$$\phi = Ux + \frac{\mu\cos\theta}{r^2}$$

$$= Ur\cos\theta + \frac{\mu\cos\theta}{r^2}$$

$$= (Ur + \mu r^{-2})\cos\theta \longrightarrow (1)$$

The velocity component at P are

$$q_r = -\frac{\partial\phi}{\partial r} = -\left(U - \frac{2\mu}{r^3}\right)\cos\theta \longrightarrow (2)$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{1}{r}(Ur + \mu r^{-2})(-\sin\theta)$$

$$= \cancel{\ } U + \frac{\mu}{r^3} \longrightarrow (3)$$

$$q_\phi = -\frac{1}{r\sin\theta}\cdot\frac{\partial\phi}{\partial\psi} = -\frac{1}{r\sin\theta}(0) = 0 \longrightarrow (4)$$

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From (2), Also $q_r = 0$

$$-\left[U - \frac{2M}{r^3}\right]\cos\theta = 0$$

$$U - \frac{2M}{r^3} = 0$$

$$U = \frac{2M}{r^3}$$

$$\boxed{r = \left(\frac{2M}{U}\right)^{1/3}}$$

$$-\cos\theta = 0$$

$$\cos\theta = 0$$

$$\theta = \cos^{-1}(0)$$

$$\boxed{\theta = \frac{\pi}{2}}$$

It follows that there is no flow over the surface of the sphere $r = a$

[$\because q_r = 0$ at $r = a$

In $r \geq a$, with $U - \frac{2M}{a^3} = 0$

$$\frac{2M}{a^3} = U$$

$$M = \frac{Ua^3}{2}$$

Sub. in eqn. (2), we get, the same velocity potential as was obtained for uniform flow past a stationary sphere of radius 'a'.

The region $r \geq a$,

the analysis of this sphere problem is the same as that of the dipole of the sphere $\frac{1}{2}Ua^3$ in the uniform stream of velocity $-U\mathbf{i}$, the axis of the dipole being in the direction of $\mathbf{i}$.

Example

1. Illustration of line distribution

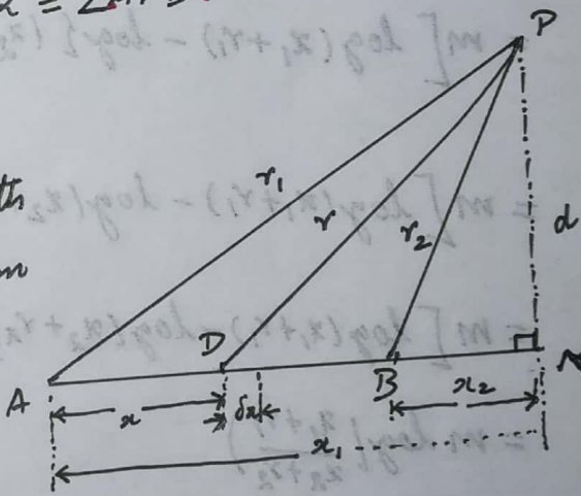
Prove that the velocity potential at a point P due to a uniform finite line source AB of strength m per unit length is of the form $\phi = m \log f$, where

$$f = \frac{r_2 + x_2}{r_1 + x_1} = \frac{r_1 - x_1}{r_2 - x_2} = \frac{a+l}{a-l}$$

in which $AB = 2l$, $PA = r_1$, $PB = r_2$, $NA = x_1$, $NB = x_2$, N being the foot of the perpendicular form P on the line AB, and $2a$ the length of the major axis of the spheroid through P having A, B as foci.

Show that the velocity at P is $\frac{(2ml\cos\alpha)}{(a^2 - l^2)u}$, where u is the unit vector along the normal to the spheroid at P and $2\alpha = \angle APB$.

Sol. ~~Let P be any point on~~



The line section of length δx in AB at distance x from A is effectively a point source of strength $m\,\delta x$ giving a velocity potential at P of amount $\frac{m\,\delta x}{r}$

where $AP = r_1$; $AN = x_1$, $BN = x_2$, where AN is the orthogonal projection of AP on AB. Also let $PN = d$, $AP = r_1$, $BP = r_2$.

From elementary geometry,

$PD^2 = DN^2 + PN^2$

$r^2 = (x_1 - x)^2 + d^2$

$= (x_1 - x)^2 + (r_1^2 - x_1^2)$

The total Velocity Potential at P due to the entire line distribution AB (=2l) is

$$\phi = \int_0^{2l} \frac{m\,dx}{r} \longrightarrow (1)$$

$$\phi = m\int_0^{2l} \frac{dx}{\sqrt{(x_1-x)^2+(r_1^2-x_1^2)}} \qquad \left[\because \int_\alpha^\beta \frac{1}{\sqrt{x^2+a^2}}\,dx = \left[\log\left(x+\sqrt{x^2+a^2}\right)\right]_\alpha^\beta\right.$$

$$= m\left[\frac{\log\left\{(x_1-x)+\sqrt{(x_1-x)^2+(r_1^2-x_1^2)}\right\}}{-1}\right]_0^{2l}$$

$$= m\left[\log\left\{(x_1-x)+\sqrt{(x_1-x)^2+(r_1^2-x_1^2)}\right\}\right]_{2l}^0$$

$$= m\left[\log\left\{x_1+\sqrt{\cancel{x_1^2}+r_1^2-\cancel{x_1^2}}\right\}-\log\left\{(x_1-2l)+\sqrt{(x_1-2l)^2+(r_1^2-x_1^2)}\right\}\right]$$

$$= m\left[\log(x_1+r_1)-\log\left\{x_2+\sqrt{x_2^2+r_1^2-x_1^2}\right\}\right]$$

$$(\because x_1-2l = x_1-AB = x_2)$$

$$= m\left[\log(x_1+r_1)-\log\left(x_2+\sqrt{r_2^2}\right)\right]$$

$$= m\left[\log(x_1+r_1)-\log(x_2+r_2)\right] \qquad \text{where } r_1^2-x_1^2=d^2=r_2^2-x_2^2$$

$$= m\log\left(\frac{x_1+r_1}{x_2+r_2}\right)$$

Again, the relation $r_1^2-x_1^2=r_2^2-x_2^2$

$$\Rightarrow (r_1-x_1)(r_1+x_1)=(r_2-x_2)(r_2+x_2)$$

$$\Rightarrow \frac{r_1+x_1}{r_2+x_2}=\frac{r_2-x_2}{r_1-x_1}=\frac{r_1+r_2+x_1-x_2}{r_1+r_2-x_1+x_2}$$

$$=\frac{r_1+r_2+2l}{r_1+r_2-2l}$$

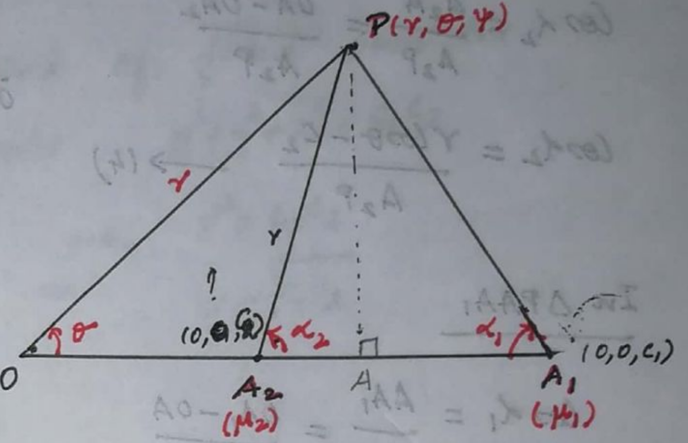
Thus, $\phi = m\log\left(\dfrac{r_1+r_2+2l}{r_1+r_2-2l}\right) = m\log\left(\dfrac{a+l}{a-l}\right)$

Where $2a$, is the length of major axis of the ellipsoid of revolution through P having A and B as foci since for such an ellipsoid.

$$r_1 + r_2 = \text{constant}$$

It follows from here that the equipotential surfaces $\phi = \text{constant}$ are precisely the family of confocal ellipsoid $r_1 + r_2 = 2a$ obtained when 'a' is allowed to vary.

2). Doublet of strength μ_1, μ_2 are situated at point A_1, A_2 whose Cartesian coordinates are $(0,0,c_1)$, $(0,0,c_2)$ their axes being ~~direction~~ directed towards and away from the origin respectively. Find the condition that there is no transport of fluid over the surface of the sphere $x^2+y^2+z^2=c_1c_2$.



Velocity potential due to doublet

$$\phi = \frac{\mu\cos\theta}{r^2}$$

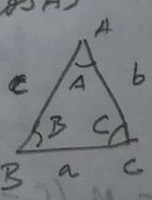
Let OA_2A_1 be the initial line

Let $P(r,\theta,\psi)$ has spherical polar coordinates & be any point in the fluid.

The axes of the doublets at A_1, A_2 make angles α_1, α_2 with A_1P, A_2P. Then the velocity potential at P is

$$\phi = \frac{\mu_2\cos\alpha_2}{A_2P^2} + \frac{\mu_1\cos\alpha_1}{A_1P^2} \longrightarrow (1)$$

By applying cosine formula $(a^2=b^2+c^2-2bc\cos A)$



In ΔOA_1P

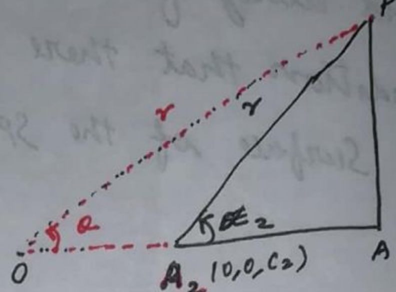
$$(A_1P)^2 = (OP)^2+(OA_1)^2-2(OP)(OA_1)\cos\theta$$

$$(A_1P)^2 = r^2+c_1^2-2rc_1\cos\theta \longrightarrow (2)$$

In ΔOA_2P

$$(A_2P)^2 = (OP)^2 + (OA_2)^2 - 2(OP)(OA_2)\cos\theta$$

$$(A_2P)^2 = r^2 + c_2^2 - 2rc_2\cos\theta \longrightarrow (3)$$



In ΔPAA_2

$$\cos\alpha_2 = \frac{A_2A}{A_2P} = \frac{OA - OA_2}{A_2P}$$

$$\cos\alpha_2 = \frac{r\cos\theta - c_2}{A_2P} \longrightarrow (4)$$

$$\cos\theta = \frac{OA}{OP}$$

$$OA = OP \cdot \cos\theta$$

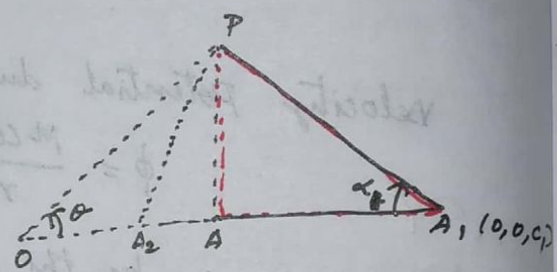
$$OA = r \cdot \cos\theta$$

$$\cos\theta = \frac{OA}{OP}$$

In ΔPAA_1

$$\cos\alpha_1 = \frac{AA_1}{A_1P} = \frac{OA_1 - OA}{A_1P}$$

$$= \frac{c_1 - r\cos\theta}{A_1P} \longrightarrow (5)$$



Sub. (2), (3), (4) & (5) in (1) we get

$$\phi = \frac{M_1(c_1 - r\cos\theta)}{(A_1P)(r^2 + c_1^2 - 2rc_1\cos\theta)} + \frac{M_2(r\cos\theta - c_2)}{(A_2P)(r^2 + c_2^2 - 2rc_2\cos\theta)}$$

$$= \frac{M_1(c_1 - r\cos\theta)}{(r^2 + c_1^2 - 2rc_1\cos\theta)^{1/2}(r^2 + c_1^2 - 2rc_1\cos\theta)^1}$$

$$+ \frac{M_2(r\cos\theta - c_2)}{(r^2 + c_2^2 - 2rc_2\cos\theta)^{1/2}(r^2 + c_1^2 - 2rc_1\cos\theta)^1}$$

$$\phi = M_1(c_1 - r\cos\theta)(r^2 + c_1^2 - 2rc_1\cos\theta)^{-3/2}$$

$$+ M_2(r\cos\theta - c_2)(r^2 + c_2^2 - 2rc_2\cos\theta)^{-3/2}$$

$$\frac{\partial\phi}{\partial r} = \mu_1 \Big\{ (c_1 - r\cos\theta)\left(-\tfrac{3}{2}\right)(r^2+c_1^2-2rc_1\cos\theta)^{-5/2}$$
$$(2r-2c_1\cos\theta) + (r^2+c_1^2-2rc_1\cos\theta)^{-3/2}(-\cos\theta)\Big\}$$

$$+ \mu_2\Big\{ (r\cos\theta - c_2)\left(-\tfrac{3}{2}\right)(r^2+c_2^2-2rc_2\cos\theta)^{-5/2}$$
$$(2r-2c_2\cos\theta) + (r^2+c_2^2-2rc_2\cos\theta)^{-3/2}\cos\theta\Big\}$$

There is no transport of fluid over the surface of the sphere, $x^2+y^2+z^2 = c_1c_2$

i.e, $r^2 = c_1c_2$

$r = \sqrt{c_1c_2}$

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=\sqrt{c_1c_2}} = 0$$

$$\Big[\mu_1\Big\{(c_1 - r\cos\theta)\left(-\tfrac{3}{2}\right)(r^2+c_1^2-2rc_1\cos\theta)^{-5/2}(2r-2c_1\cos\theta)$$
$$(r^2+c_1^2-2rc_1\cos\theta)^{-3/2}\cos\theta\Big\}$$
$$+\mu_2\Big\{(r\cos\theta-c_2)\left(-\tfrac{3}{2}\right)(r^2+c_2^2-2rc_2\cos\theta)^{-5/2}(2r-2c_2\cos\theta)$$
$$+(r^2+c_2^2-2rc_2\cos\theta)^{-3/2}\cos\theta\Big\}\Big] = 0$$

at $r=\sqrt{c_1c_2}$

$$-\mu_1\Big\{(c_1-\sqrt{c_1c_2}\cos\theta)\left(-\tfrac{3}{2}\right)(c_1c_2+c_1^2-2c_1^{3/2}c^{1/2}(\cos\theta))^{-5/2}(2)(c_1^{1/2}c_2^{1/2}-c_1\cos\theta)$$
$$-(c_1c_2+c_1^2-2c_1^{3/2}c_2^{1/2}\cos\theta)^{-3/2}\cos\theta\Big\}$$
$$+\mu_2\Big\{(c_1^{1/2}c_2^{1/2}\cos\theta-c_2)\left(-\tfrac{3}{2}\right)(c_1c_2+c_2^2-2rc_2\cos\theta)^{-5/2}(2)(c_1^{1/2}c_2^{1/2}-c_2\cos\theta)$$
$$+(c_1c_2+c_2^2-2c_2^{3/2}c_1\cos\theta)^{-3/2}\cos\theta\Big\} = 0$$

$$\Rightarrow \mu_2\Big[\cos\theta\{c_1c_2-2(c_1c_2)^{1/2}\cos\theta+c_2^2\}^{-3/2} - 3\{(c_1c_2)^{1/2}\cos\theta-c_2\}$$
$$\{c_1c_2-2c_2(c_1c_2)^{1/2}\cos\theta+c_2^2\}^{-5/2} + \{(c_1c_2)^{1/2}-c_2\cos\theta\}\Big]$$
$$=\mu_1\Big[\cos\{c_1c_2-2(c_1c_2)^{1/2}c_1\cos\theta+c_1^2\}^{-3/2}+3\{c_1-(c_1c_2)^{1/2}\cos\theta\}$$
$$\{c_1c_2-2c_1(c_1c_2)^{1/2}\cos\theta+c_1^2\}^{-5/2} + \{c_1-(c_1c_2)^{1/2}\cos\theta\}\Big]$$

$$\frac{M_2}{M_1} = \frac{\cos\theta\, c_1^{-3/2} \left[c_2 - 2\sqrt{c_1 c_2}\cos\theta + c_1 \right)^{-3/2} + 3c_2^{1/2}(c_2^{1/2} - c_1^{1/2}\cos\theta)\, c_1^{-5/2}(c_2 - 2\sqrt{c_1c_2})\cos\theta + c_2)^{5/2} \cdot c_1^{1/2}(c_2^{1/2} - c_1^{1/2}\cos\theta)}{\cos\theta\, c_2^{-3/2}(c_1 - 2\sqrt{c_1c_2}\cos\theta + c_2)^{-3/2} + 3c_2^{1/2}(c_2^{1/2} - c_1^{1/2}\cos\theta)\, c_2^{-5/2}(c_1 - 2\sqrt{c_1c_2}\cos\theta + c_2)^{5/2}\, c_2^{1/2}}$$

$$\Rightarrow \frac{M_2}{M_1} = \frac{c_1^{-3/2}}{c_2^{-3/2}}$$

$$\Rightarrow \frac{M_2}{M_1} = \left(\frac{c_2}{c_1}\right)^{3/2}$$

1. Prove that the velocity potential at a point P due to a uniform finite line source AB of strength m per unit length is of the form $\phi = m \log f$, where

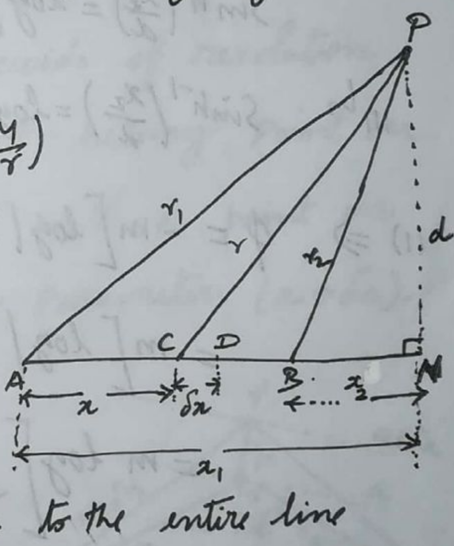
$$f = \frac{r_2 + x_2}{r_1 + x_1} = \frac{r_1 - x_1}{r_2 - x_2} = \frac{a+l}{a-l}$$

in which $AB = 2l$, $PA = r_1$, $PB = r_2$, $NA = x_1$, $NB = x_2$, N being the foot of the perpendicular from P on the line AB, and $2a$ the length of the major axis of the spheroid through P having A, B as foci.

Show that the velocity at P is $\frac{2ml\cos\alpha}{a^2 - l^2}\hat{u}$, where u is the unit vector along the normal to the spheroid at P and $2\alpha = \angle APB$

Sol.: The line section of length $CD = \delta x$ in AB at distance x from A is effectively a point source $m\delta x$ giving a velocity potential at P is

$$\phi = \frac{m\delta x}{r} \qquad \left(\because \phi = \frac{M}{r}\right)$$



In ΔCPN

$$r^2 = d^2 + CN^2$$
$$= d^2 + (AN - AC)^2$$
$$= d^2 + (x_1 - x)^2$$

The total velocity potential at P due to the entire line distribution is

$$\phi = \int_0^{2l} \frac{m\,dx}{r} = m\int_0^{2l} \frac{dx}{\sqrt{d^2 + (x_1 - x)^2}} = -m\left[\sinh^{-1}\left(\frac{x_1 - x}{d}\right)\right]_0^{2l}$$

$$= -m\left[\sinh^{-1}\left(\frac{x_1 - 2l}{d}\right) - \sinh^{-1}\left(\frac{x_1}{d}\right)\right]$$

$$= -m\left[\sinh^{-1}\left(\frac{x_2}{d}\right) - \sinh^{-1}\left(\frac{x_1}{d}\right)\right] \longrightarrow (1)$$

Consider $\sinh^{-1}(x_2/d) = y$,

$$\frac{x_2}{d} = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$\frac{2x_2}{d} = e^y - e^{-y}$$

$$\frac{2x_2}{d} = e^y - \frac{1}{e^y}$$

$$\frac{2x_2 e^y}{d} = e^{2y} - 1$$

$$e^{2y} - \frac{2x_2}{d} e^y - 1 = 0$$

$$e^y = \frac{\frac{2x_2}{d} \pm \sqrt{\frac{4x_2^2}{d^2} - 4(1)(-1)}}{2(1)}$$

$$= \frac{\cancel{2}x_2 \pm \cancel{2}\sqrt{x_2^2+d^2}}{\cancel{2}d}$$

Take log,

$$y = \log\left[\frac{x_2 \pm \sqrt{x_2^2+d^2}}{d}\right]$$

$$\sinh^{-1}\left(\frac{x_2}{d}\right) = \log\left(\frac{x_2+\sqrt{x_2^2+d^2}}{d}\right)$$

$$\text{1}^{ly} \quad \sinh^{-1}\left(\frac{x_1}{d}\right) = \log\left(\frac{x_1+\sqrt{x_1^2+d^2}}{d}\right)$$

$$(1) \Rightarrow \quad \phi = -m\left[\log\left(\frac{x_2+\sqrt{x_2^2+d^2}}{d}\right) - \log\left(\frac{x_1+\sqrt{x_1^2+d^2}}{d}\right)\right]$$

$$= m\left[\log\left(\frac{x_1+\sqrt{x_1^2+d^2}}{d}\right) - \log\left(\frac{x_2+\sqrt{x_2^2+d^2}}{d}\right)\right]$$

$$= m\log\left[\frac{\frac{x_1+\sqrt{x_1^2+d^2}}{\cancel{d}}}{\frac{x_2+\sqrt{x_2^2+d^2}}{\cancel{d}}}\right] = m\log\left[\frac{x_1+\sqrt{x_1^2+d^2}}{x_2+\sqrt{x_2^2+d^2}}\right]$$

$$= m\log\left[\frac{x_1+\sqrt{r_1^2}}{x_2+\sqrt{r_2^2}}\right] = m\log\left(\frac{x_1+r_1}{x_2+r_2}\right)$$

Now, $r_1^2 - x_1^2 = r_2^2 - x_2^2 = d^2$

$$(r_1-x_1)(r_1+x_1) = (r_2-x_2)(r_2+x_2) = d^2$$

$$\frac{r_1+x_1}{r_2+x_2} = \frac{r_2-x_2}{r_1-x_1} = \frac{r_1+x_1+r_2-x_2}{r_2+x_2+r_1-x_1} = \frac{r_1+r_2+2l}{r_1+r_2-2l}$$

At P on the Spheroid through P having A, B as foci

$r_1 + r_2 = 2a$

Hence $\dfrac{r_1+x_1}{r_2+x_2} = \dfrac{r_2-x_2}{r_1-x_1} = \dfrac{2a+2l}{2a-2l} = \dfrac{a+l}{a-l}$

$$\phi = m\log\left(\frac{r_1+x_1}{r_2+x_2}\right) = m\log\left(\frac{a+l}{a-l}\right) \quad \text{———} (*)$$

The equipotentials are given by $\phi = \text{constant}$

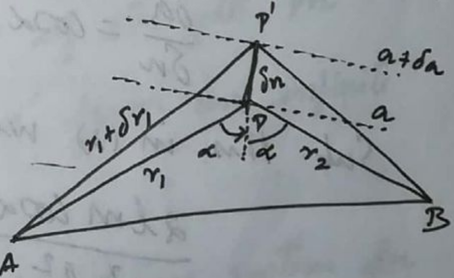
$$m\log\left(\frac{a+l}{a-l}\right) = \text{constant}$$

$$\frac{a+l}{a-l} = \text{constant} \quad \Rightarrow \quad a = \text{constant}$$

i.e. $\dfrac{r_1+r_2}{2} = a = \text{constant}$

$$r_1 + r_2 = \text{Constant}$$

These Surface are confocal ellipsoids of revolution about AB with A, B as foci. Let P be any point on the ellipsoid specified by parameters a, P' a point on the neighbouring ellipsoid specified by parameter $(a+\delta a)$, where $\overline{PP'} \equiv \delta n u$.



Then the velocity at P is

$$\bar{q} = -\nabla\phi = -\frac{\partial \phi}{\partial n}\hat{u}$$

$$= -\frac{\partial}{\partial n}\left\{m\log\left(\frac{a+l}{a-l}\right)\right\}\hat{u}$$

$$= -m\frac{\partial}{\partial n}\left\{\log(a+l) - \log(a-l)\right\}\hat{u}$$

$$= -m\hat{u}\left\{\frac{1}{a+l} - \frac{1}{a-l}\right\}\frac{\partial a}{\partial n}$$

$$-\nabla\phi = -m\hat{u}\left\{\frac{a-l-a-l}{(a+l)(a-l)}\right\}\frac{\partial a}{\partial n}$$

$$-\nabla\phi = -m\hat{u}\left(\frac{-2l}{a^2-l^2}\right)\frac{\partial a}{\partial n} = \frac{2lm}{a^2-l^2}\frac{\partial a}{\partial n}\hat{u} \quad \longrightarrow (2)$$

Now the normal at P bisects 2α, the angle between the focal radii AP, PB.

From $\Delta APP'$

$$(r_1+\delta r_1)^2 = r_1^2 + (\delta n)^2 - 2r_1\,\delta n \cos(\pi-\alpha)$$

$$= r_1^2 + \delta n^2 + 2r_1\,\delta n \cos\alpha$$

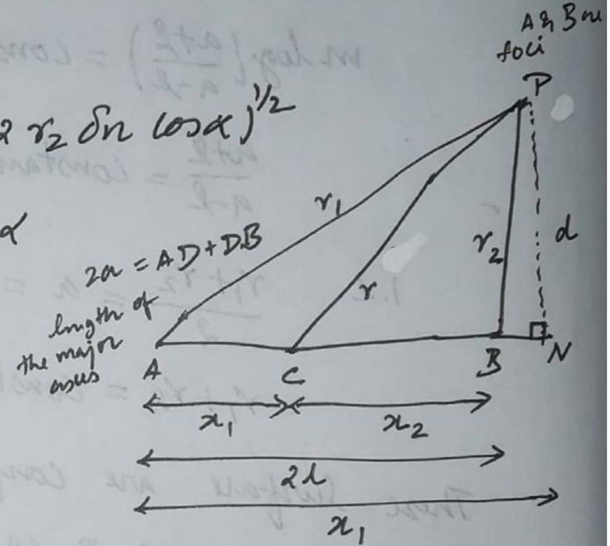
$$(r_1+\delta r_1) = (r_1^2+\delta n^2+2r_1\delta n\cos\alpha)^{1/2}$$

$$\simeq r_1 + \delta n\cos\alpha$$

|||ly from $\Delta BPP'$

$$(r_2+\delta r_2) = (r_2^2+\delta n^2+2r_2\,\delta n\cos\alpha)^{1/2}$$

$$r_2+\delta r_2 \simeq r_2 + \delta n\cos\alpha$$



Adding we get

$$r_1+\delta r_1+r_2+\delta r_2 = r_1+r_2+\delta n\cos\alpha + \delta n\cos\alpha$$

$$\cancel{r_1}+\delta r_1+\cancel{r_2}+\delta r_2-\cancel{r_1}-\cancel{r_2} = 2\delta n\cos\alpha$$

$$\delta r_1+\delta r_2 = 2\delta n\cos\alpha$$

$$\cancel{2}\delta a = \cancel{2}\delta n\cos\alpha$$

$$\frac{\delta a}{\delta n} = \cos\alpha$$

Sub. this in (2) we get

$$\frac{2lm\cos\alpha}{a^2-l^2}\,\hat{u}$$

Hence the velocity at P is $\dfrac{2lm\cos\alpha}{a^2-l^2}\,\hat{u}$

—x—

Images in a Rigid infinite plane

Suppose a surface S can be drawn in a moving fluid in such a way that there is no transport of fluid across S. Let S divide the fluid into two regions labelled 1, 2. Then any system of the Sources, Sinks (or) doublets in 2 is called an image system of the region 1 in S.

If we remove the fluid in 2 and replace S by a rigid boundary of the same size and shape, then the flow in 1 is unaltered in accordance with the conditions of the uniqueness theorems.

It follows, then, that if we know the image system for 1 in S, we can solve the problem of flow in 1 against a rigid surface S.

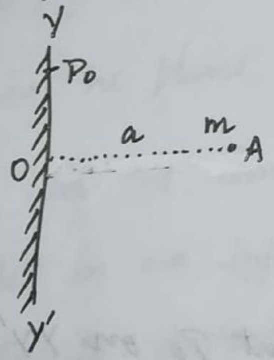


Fig (i)

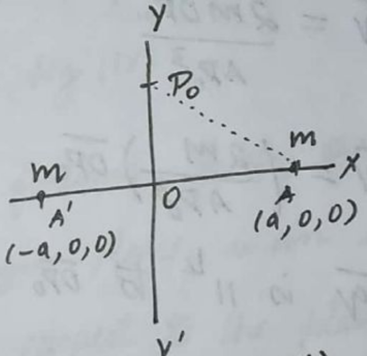


Fig (ii)

Fig (i) Shows a simple source of strength 'm' situated m situated at a distance 'a' from an infinite rigid plane YY'.

We first show that the appropriate image system for this is an equal source at A', the optic image of A in the plane.

Consider fig (ii) in which we have equal sources of strength m a A(a,0,0) and A'(-a,0,0).

Let P_0 be any point on the plane YY' in fig (ii) then the fluid velocity at P_0 is,

$$\bar{q} = \frac{m}{AP_0^2} \cdot \hat{AP_0} + \frac{m}{A'P_0^2} \cdot \hat{A'P_0}$$

$$= \frac{m}{AP_0^2} \cdot \hat{AP_0} \cdot \frac{AP_0}{AP_0} + \frac{m}{A'P_0^2} \cdot \hat{A'P_0} \cdot \frac{A'P_0}{A'P_0}$$

$$= \frac{m \, \overline{AP_0}}{AP_0^3} + \frac{m}{A'P_0^3} \cdot \overline{A'P_0}$$

$$= \frac{m}{AP_0^3} \left(\overline{AP_0} + \overline{A'P_0} \right)$$

$$= \frac{m}{AP_0^3} \left(\overline{AO} + \overline{OP_0} + \overline{A'O} + \overline{OP_0} \right)$$

$$= \frac{m}{AP_0^3} \left(\cancel{a} + \overline{OP_0} - \cancel{a} + \overline{OP_0} \right)$$

$$\bar{q} = \frac{2m\,\overline{OP_0}}{AP_0^3}$$

$$\bar{q} = \left(\frac{2m}{AP_0^3} \right) \overline{OP_0}$$

$$\Rightarrow \bar{q} \text{ is } \parallel^{le} \text{ to } \overline{OP_0}$$

This shows that at any point P_0 on YY', the fluid flows tangentially to the plane YY'. Thus there is no transport of fluid across this plane.

Thus in fig (ii) at all corresponding points P_0 on the surfaces YY'

$$\bar{q} \cdot \hat{n} = 0$$

$$-\nabla\phi \cdot \hat{n} = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial n} = 0 \text{ for the region of flow } x \geq 0$$

By uniqueness theorem, that the image of m at A in YY' in Fig (i) is m at A', the optic image of A in

YY'.

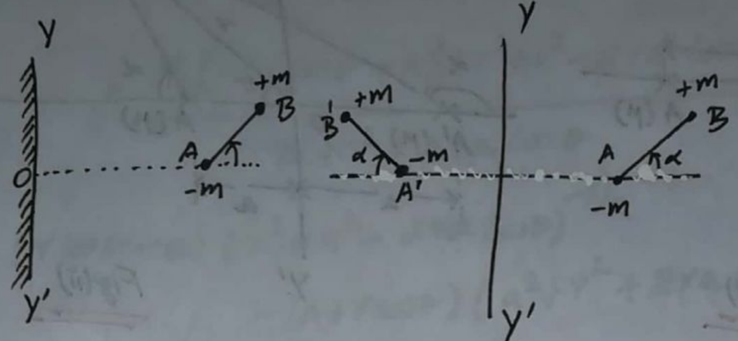


Fig. (iii) fig (iv)

Consider a pair of Sources $-m$ at A, m at B close together and on one side of the rigid plane YY' in fig (iii).

The image system is $-m$ at A', m at B', where A', B' are the respective optic images of the point A, B in the plane YY' in fig (iv).

In the limit it follows, then, the image of a doublet in an infinite rigid plane is an equal doublet symmetrically disposed with respect to the plane.

Example

1) A three-dimensional doublet of strength μ whose axis is in the direction $\overline{Ox}$ is distance 'a' from the rigid plane $x=0$ which is the sole boundary of liquid of density ρ, infinite in extend. Find the pressure at a point on the boundary distant r from the doublet given that the pressure at infinite is P_∞. Show that the pressure on the plane is least at a distance $a\sqrt{5}/2$ from doublet.